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Unitary representations, branching rules and matrix elements for the non-compact symplectic groups

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Abstract. The complementarity of the symplectic and orthogonal groups is used to infer properties of the infinite-dimensional unirreps of the former from the character theory of the latter. The complete set of D_+-series metaplectic unirreps of Sp(N, R) is identified and branching rules are given for their restrictions to the maximal compact subgroup, U(N), developed in terms of the properties of Schur functions. A known algorithm for the evaluation of matrix elements of the Sp(3, R) Lie algebra is extended to any Sp(N, R) and analytic expressions are given for important classes of unirreps and multiplicity free states.

1. Introduction

The development of algorithms for evaluating Kronecker products and branching rules for the various compact Lie groups has been the subject of many studies (cf Murnaghan 1938, Weyl 1939, Littlewood 1940, Racah 1964, Hamermesh 1962, Judd 1963, Wybourne 1970, Vanagas 1971, King 1975, Black *et al* 1983, Black and Wybourne 1983 and references therein).

Rather less is known about the non-compact symplectic groups which have been used extensively in recent times in the theory of nuclear collective motion (Arickx *et al* 1979, Rosensteel and Rowe 1977, 1980, Park *et al* 1984).

Of major importance in character theory is the fact that the general linear group, GL(N), and the symmetric group, S_n, are complementary in their actions on the tensors of an N-dimensional vector space. As a result the characters of the two groups are related and it is possible to understand many of their properties in terms of the well developed algebra of Schur functions (cf Littlewood 1940, Macdonald 1979).

Moshinsky and Quesne (1971) (also Kashiwara and Vergne 1978) pointed out a similar complementarity of the Sp(N, R) and O(n) actions on the states of the Nn-dimensional harmonic oscillator (or equivalently, in the Bargmann (1961) representation, on polynomials in Nn variables). Whereas the GL(N)$\times S_n$ complementarity relates the unirreps of compact continuous groups to those of finite groups, we show that the Sp(N, R)$\times$O(n) complementarity relates those of non-compact groups to those of compact groups.

Character theory has primarily been used for compact groups because the non-trivial unirreps of a non-compact group are of infinite dimension. We show here that Sp(N, R)

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characters are expandable as well defined infinite series of U(N) characters (or equivalently as Schur functions) and expressible in terms of already familiar infinite series (Littlewood 1940, King 1975, Black *et al* 1983).

We restrict consideration to the positive discrete (D_+) series representations of the two-fold covering (metaplectic) groups. The D_+ unirreps of the Sp(N, R) groups were identified by Godement (1958) and elaborated on by Rosensteel and Rowe (1977, 1980).

Matrix elements of the Sp(1, R) algebra have been known for some time (e.g. Barut 1967). Matrix elements for Sp(3, R) have been calculated numerically (Rosensteel 1980, Rosensteel and Rowe 1983). Recently Castaños *et al* (1984) gave analytic expressions for the associated class of ($\sigma_1 = \sigma_2 = \sigma_3$) unirreps of Sp(3, R). Rowe *et al* (1984), using the coherent state theory of Rowe (1984), then gave analytic matrix elements for any Sp(3, R) unirrep whose Sp(3, R)$\downarrow$U(3) branching is multiplicity free. More generally Rowe (1984) gave an algorithm for the matrix elements for any D_+ unirrep of Sp(3, R). In this paper we give the natural extension of these results to any Sp(N, R) algebra and use them to obtain analytic matrix elements for the $\sigma(1^r)$, $r = 0, 1, 2, \ldots$ representations (to be defined in § 3).

2. The Sp(N, R) algebra

The Sp(N, R) group is fundamentally the group of linear canonical transformations of a $2N$-dimensional phase space.

A convenient basis for C_N, the complex extension of its Lie algebra, is a set

$$\{A_{ij}, B_{ij}, C_{ij}; i, j = 1, \ldots, N\} \tag{2.1}$$

where $A_{ij} = A_{ji}$, $B_{ij} = B_{ji}$ are symmetric, where (C_{ij}) is a basis for the U(N) subalgebra, and which satisfy the commutation relations

$$\begin{aligned} [C_{ij}, C_{lk}] &= \delta_{jl}C_{ik} - \delta_{ik}C_{lj} \\ [C_{ij}, A_{lk}] &= \delta_{jl}A_{ik} + \delta_{jk}A_{il} \\ [B_{lk}, C_{ij}] &= \delta_{ki}B_{lj} + \delta_{li}B_{kj} \\ [B_{ij}, A_{lk}] &= \delta_{il}C_{kj} + \delta_{ik}C_{lj} + \delta_{jl}C_{ki} + \delta_{jk}C_{li}. \end{aligned} \tag{2.2}$$

A possible realisation of this basis, that exhibits Sp(N, R) as the dynamical group of the harmonic oscillator, is given by

$$A_{ij} = b_i^\dagger b_j^\dagger, \qquad B_{ij} = b_i b_j, \qquad C_{ij} = \tfrac{1}{2}(b_i^\dagger b_j + b_j b_i^\dagger).$$

This realisation is too restrictive, however, for our purposes.

Following Rosensteel and Rowe (1980), a discrete series representation $\tau = (\tau_1, \tau_2, \ldots, \tau_N)$ is defined by a lowest weight state $|\tau Lw\rangle$ satisfying

$$\begin{aligned} &\gamma(C_{ii})|\tau LW\rangle = \tau_i|\tau LW\rangle \\ &\gamma(C_{ij})|\tau LW\rangle = 0 \qquad i<j \\ &\gamma(B_{ij})|\tau LW\rangle = 0 \qquad \text{for all } i, j \end{aligned} \tag{2.3}$$

where γ gives the action on the state space. The space of states is generated by the raising operators as usual. Note that $|\tau LW\rangle$ is also a lowest weight state for the U(N) subalgebra.

Instead of U(N) labels (τ), it is more convenient to use labels $\sigma(\lambda)$ where

$$\tau_i = \sigma + \lambda_i \qquad \text{for } i = 1, 2, \ldots, N \tag{2.4}$$

with σ chosen such that

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N \geqslant 0 \tag{2.5}$$

is a regular partition of integers. Note that among the set of equivalent labels $\sigma(\lambda)$ for a representation (τ), there is one $\sigma^{\max}(\lambda^{\min})$ for which $\sigma^{\max} = \tau_N$ and $\lambda_N^{\min} = 0$.

In order that the representation $\sigma(\lambda)$ of the algebra integrates to a representation of the Sp(N, R) group, σ must be restricted to integer values (Godement 1958). We consider here also the representation of the two-fold covering (metaplectic) group, for which σ can take half-integer values. However, we restrict to $\sigma^{\max} \geqslant 0$, since as one can easily show, every representation with $\sigma^{\max} < 0$ is contragredient to another with $\sigma^{\max} > 0$. The $\sigma^{\max} > 0$ representations constitute the D_+ series. Note that for unitarity, all matrix elements must satisfy

$$\langle\alpha|\gamma(B_{ij})|\beta\rangle\langle\beta|\gamma(A_{ij})|\alpha\rangle \geqslant 0. \tag{2.6}$$

The above unitarity condition constrains the labels $\sigma(\lambda)$ in accordance with the following important theorem, to be proved in § 5, which confirms a conjecture by Kashiwara and Vergne (1978).

Theorem 1. The Sp(N, R) representation $\sigma(\lambda)$ is unitary if and only if

$$\tilde{\lambda}_1^{\min} \leqslant N - 1 \tag{2.7}$$

and

$$\tilde{\lambda}_1^{\min} + \tilde{\lambda}_2^{\min} \leqslant 2\sigma^{\max} \tag{2.8}$$

where $(\tilde{\lambda}) = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots)$ is the partition conjugate to (λ). Thus $\tilde{\lambda}_i$ is the number of parts λ_j of (λ) with $\lambda_j \geqslant i$.

We will describe representations that satisfy this constraint as 'admissible'. Associated with every integer or half-integer value of $\sigma^{\max}$ there will be a set of admissible representations of the D_+ series of Sp(N, R) labelled as $\sigma(\lambda)$.

The characters for the admissible unirreps of the D_+ series of Sp(N, R) will be designated as $\langle\sigma(\lambda)\rangle$. Such a character labelling constitutes a 'natural' labelling scheme (Wybourne and Bowick 1977, King and Al-Qubanchi 1981) and, under Sp(N, R)$\downarrow$U(N), $\langle\sigma(\lambda)\rangle$ restricts to an infinite sum of U(N) characters of which the leading term is given by

$$\langle\sigma(\lambda)\rangle \downarrow \varepsilon^{\sigma}\{\lambda\} + \ldots \tag{2.9}$$

where ε is just the one-dimensional character $\{1^N\}$ of U(N) given by the determinant of the group element. Our objective is to find the subsequent terms of the sequence (2.9).

3. The Sp(N, R) $\downarrow$ U(N) branching rules

According to the complementarity theorem of Moshinsky and Quesne (1971) and Kashiwara and Vergne (1978) the states of the $2\sigma N$-dimensional harmonic oscillator that belong to an O(2σ) unirrep also belong to an Sp(N, R) unirrep. Furthermore, the irreps of the product group Sp(N, R) $\times$ O(2σ) always occur without multiplicity.

In terms of characters this implies (Kashiwara and Vergne 1978) a branching rule for the two fundamental unirreps $\langle 1/2(0)\rangle$ and $\langle 1/2(1)\rangle$ of $\mathrm{Sp}(2\sigma N, R)$ under

$$\mathrm{Sp}(2\sigma N, R)\downarrow \mathrm{Sp}(N, R)\times \mathrm{O}(2\sigma)$$
$$\langle 1/2(0)\rangle+\langle 1/2(1)\rangle\downarrow \sum_{\lambda\in S}\langle \sigma(\lambda)\rangle\times[f(\lambda)] \tag{3.1}$$

where $[f(\lambda)]$ is the character of some unirrep of $\mathrm{O}(2\sigma)$ determined uniquely by (λ) and where S denotes the set of partitions that occur in this reduction.

The $\mathrm{U}(N)$ content of an $\mathrm{Sp}(N, R)$ unirrep $\sigma(\lambda)$ is inferred by comparing the branching rules for

$$\mathrm{Sp}(2\sigma N, R)\downarrow \mathrm{Sp}(N, R)\times \mathrm{O}(2\sigma)\downarrow \mathrm{U}(N)\times \mathrm{O}(2\sigma)$$

and

$$\mathrm{Sp}(2\sigma N, R)\downarrow \mathrm{U}(2\sigma N)\downarrow \mathrm{U}(N)\times \mathrm{O}(2\sigma).$$

The $\mathrm{U}(2\sigma N)$ content of the simple harmonic oscillator in $2\sigma N$ dimensions is well known. Under the restriction

$$\mathrm{Sp}(2\sigma N, R)\downarrow \mathrm{U}(2\sigma N)$$
$$\langle 1/2(0)\rangle+\langle 1/2(1)\rangle\downarrow \varepsilon^{1/2}M \tag{3.2}$$

where M is the infinite S-function series (Black *et al* 1983)

$$M=\sum_m \{m\}$$

summed over all non-negative integers.

The $\mathrm{U}(2\sigma N)\downarrow \mathrm{U}(N)\times \mathrm{U}(2\sigma)\downarrow \mathrm{U}(N)\times \mathrm{O}(2\sigma)$ branching rules (King 1975) give:

$$\varepsilon^{1/2}\downarrow \varepsilon^{\sigma}\times\varepsilon^{N/2}\downarrow \varepsilon^{\sigma}\times(\pm 1)^{N/2}$$
$$M\downarrow \sum_{\zeta}\{\zeta\}\times\{\zeta\}\downarrow \sum_{\zeta}\{\zeta\}\times[\zeta/D] \tag{3.3a}$$

where

$$D=\{2\}\otimes M=\sum_{\delta}\{\delta\}$$

is the infinite series (Black *et al* 1983) of S-functions for which each partition (δ) only involves parts which are even. From the definition of the S-function quotient (Littlewood 1940, p 108) and its relation with S-function products it follows that (3.3a) can be re-expressed as

$$M\downarrow \sum_{\zeta}\{\zeta\}\times\{\zeta\}\downarrow \sum_{\rho}\{\rho D\}\times[\rho]. \tag{3.3b}$$

This is a key step in deriving the $\mathrm{Sp}(N, R)\downarrow \mathrm{U}(N)$ branching rules.

Combining (3.2) and (3.3) we obtain

$$\mathrm{Sp}(2\sigma N, R)\downarrow \mathrm{U}(N)\times \mathrm{O}(2\sigma)$$
$$\langle 1/2(0)\rangle+\langle 1/2(1)\rangle\downarrow \sum_{\rho}\varepsilon^{\sigma}\{\rho D\}\times(\pm)^{N/2}[\rho]. \tag{3.4}$$

Comparing with (3.1), we obtain

$$\begin{array}{l} \mathrm{Sp}(N,R)\times\mathrm{O}(2\sigma)\downarrow\mathrm{U}(N)\times\mathrm{O}(2\sigma) \\ \sum_{\lambda\in S}\langle\sigma(\lambda)\rangle\times[f(\lambda)]\downarrow\sum_{\rho}\varepsilon^{\sigma}\{\rho D\}\times(\pm1)^{N/2}[\rho]. \end{array} \tag{3.5}$$

The partitions (ζ) and (ρ) in (3.3) are necessarily restricted in their number of parts by:

$$\begin{array}{l} \tilde{\rho}_1\leqslant\min(N,2\sigma) \\ \tilde{\zeta}_1\leqslant\min(N,2\sigma). \end{array} \tag{3.6}$$

However, standard labels for $\mathrm{O}(2\sigma)$ unirreps are given by partitions having not more than σ parts. Thus non-standard labels appear in the RHS of (3.5).

Those non-standard labels, which arise in the restriction $\mathrm{U}(2\sigma)\downarrow\mathrm{O}(2\sigma)$, can be related to standard labels by means of Newell's (1951) modification rules or by an equivalent method of boundary hook removals (King 1975, Black *et al* 1983).

It follows that to each standard $\mathrm{O}(2\sigma)$ label there is a sequence of equivalent non-standard labels having up to 2σ parts. We call these sequences 'signed sequences'.

A systematic procedure for deriving the signed sequences is given in § 4. The sequences for $\sigma\leqslant 5/2$ are given in table 1.

Let λ_s^{σ} denote the signed sequence having leading term (λ). It follows from (3.5) that

$$\sum_{\rho}\{\rho D\}[\rho]=\sum_{\lambda\in S}\{\lambda_s^{\sigma}D\}[\lambda]. \tag{3.7}$$

Hence we obtain

$$[f(\lambda)]=(\pm)^{N/2}[\lambda] \tag{3.8}$$

and the branching rule

$$\begin{array}{l} \mathrm{Sp}(N,R)\downarrow\mathrm{U}(N) \\ \langle\sigma(\lambda)\rangle\downarrow\varepsilon^{\sigma}\{\lambda_s^{\sigma}D\}_{2\sigma,N} \end{array} \tag{3.9}$$

where the subscript $2\sigma,N$ indicates, that in the product of S-functions, only those terms are retained for which the corresponding partition label contains no more than 2σ and no more than N parts.

It can be shown that if $\sigma\geqslant N$ or if $2\sigma>N$ and $\tilde{\lambda}_2\leqslant\max(2\sigma-N,0)$ then only the leading term (λ) in λ_s^{σ} will survive in (3.9). In such a case (3.9) simplifies to

$$\langle\sigma(\lambda)\rangle\downarrow\varepsilon^{\sigma}\{\lambda D\}_N. \tag{3.10}$$

By way of example consider the unirreps $\langle 1(\lambda_1)\rangle$ of $\mathrm{Sp}(N,R)$ with $\lambda_1\geqslant 2$. From table 1 we have the signed sequence

$$(\lambda_1)_s^1=(\lambda_1)-(\lambda_1 2)$$

and hence from (3.9) we have the general result

$$\langle 1(\lambda_1)\rangle\downarrow\varepsilon^1\{(\{\lambda_1\}-\{\lambda_1 2\})D\}_{2,N}.$$

If $N=1$ the result simplifies to

$$\langle 1(\lambda_1)\rangle\downarrow\varepsilon^1\{\lambda_1 D\}_1,$$

Table 1. Signed sequences for $\sigma \leqslant \frac{5}{2}$.

σ	(λ)	$[\lambda]$	λ_s^σ	
$\frac{1}{2}$	(0)	$[0]$	(1)	
1	(0)	$[0]$	(0)	
	(1)	$[1]$	(1)	
	(λ_1)	$[\lambda_1]$	$(\lambda_1)-(\lambda_1 2)$	$\lambda_1 \geqslant 2$
	(1^2)	$[0]^*$	(1^2)	
$\frac{3}{2}$	(0)	$[0]$	(0)	
	(1)	$[1]$	(1)	
	(λ_1)	$[\lambda_1]$	$(\lambda_1)-(\lambda_1 2^2)$	$\lambda_1 \geqslant 2$
	(1^2)	$[1]^*$	(1^2)	
	$(\lambda_1 1)$	$[\lambda_1]^*$	$(\lambda_1 1)-(\lambda_1 21)$	$\lambda_1 \geqslant 2$
	(1^3)	$[0]^*$	(1^2)	
2	(0)	$[0]$	(0)	
	(1)	$[1]$	(1)	
	(λ_1)	$[\lambda_1]$	$(\lambda_1)-(\lambda_1 2^3)$	$\lambda_1 \geqslant 2$
	(1^2)	$[1^2]$	(1^2)	
	$(\lambda_1 1)$	$[\lambda_1 1]$	$(\lambda_1 1)-(\lambda_1 2^2 1)$	$\lambda_1 \geqslant 2$
	$(\lambda_1 1)$	$[\lambda_1 1]$	$(\lambda_1 1)-(\lambda_1 2^2 1)$	$\lambda_1 \geqslant 2$
	$(\lambda_1 2)$	$[\lambda_1 2]$	$(\lambda_1 2)-(\lambda_1 2^2)$	$\lambda_1 \geqslant 2$
	$(\lambda_1 \lambda_2)$	$[\lambda_1 \lambda_2]$	$(\lambda_1\lambda_2)-(\lambda_1\lambda_2 2)+(\lambda_1\lambda_2 31)-(\lambda_1\lambda_2 3^2)$	$\lambda_1 \geqslant \lambda_2 \geqslant 3$
	(1^3)	$[1]^*$	(1^3)	
	$(\lambda_1 1^2)$	$[\lambda_1]^*$	$(\lambda_1 1^2)-(\lambda_1 21^2)$	$\lambda_1 \geqslant 2$
	(1^4)	$[0]^*$	(1^4)	
$\frac{5}{2}$	(0)	$[0]$	(0)	
	(1)	$[1]$	(1)	
	(λ_1)	$[\lambda_1]$	$(\lambda_1)-(\lambda_1 2^4)$	$\lambda_1 \geqslant 2$
	(1^2)	$[1^2]$	(1^2)	
	$(\lambda_1 1)$	$[\lambda_1 1]$	$(\lambda_1 1)-(\lambda_1 2^3 1)$	$\lambda_1 \geqslant 2$
	$(\lambda_1 2)$	$[\lambda_1 2]$	$(\lambda_1 2)-(\lambda_1 2^2)$	$\lambda_1 \geqslant 2$
	$(\lambda_1 \lambda_2)$	$[\lambda_1 \lambda_2]$	$(\lambda_1\lambda_2)-(\lambda_1\lambda_2 2^2)+(\lambda_1\lambda_2 321)-(\lambda_1\lambda_2 3^2 2)$	$\lambda_1 \geqslant \lambda_2 \geqslant 3$
	(1^3)	$[1^2]^*$	(1^3)	
	$(\lambda_1 1^2)$	$[\lambda_1 1]^*$	$(\lambda_1 1^2)-(\lambda_1 2^2 1^2)$	$\lambda_1 \geqslant 2$
	$(\lambda_1 21)$	$[\lambda_1 2]^*$	$(\lambda_1 21)-(\lambda_1 2^2 1)$	$\lambda_1 \geqslant 2$
	$(\lambda_1 \lambda_2 1)$	$[\lambda_1 \lambda_2]^*$	$(\lambda_1\lambda_2 1)-(\lambda_1\lambda_2 21)+(\lambda_1\lambda_2 31^2)-(\lambda_1\lambda_2 3^3)$	$\lambda_1 \geqslant \lambda_2 \geqslant 3$
	(1^4)	$[1]^*$	(1^4)	
	$(\lambda_1 1^3)$	$[\lambda_1]^*$	$(\lambda_1 1^3)-(\lambda_1 21^3)$	$\lambda_2 \geqslant 2$
	(1^5)	$[0]^*$	(1^5)	

while if $N \geqslant 2$ it remains as

$$\langle 1(\lambda_1)\rangle \downarrow \varepsilon^1\{(\{\lambda_1\}-\{\lambda_1 2\})D\}_2.$$

Thus for $N=1$

$$\begin{aligned}\langle 1(2)\rangle \downarrow \varepsilon^1\{2D\}_1 &= \varepsilon^1\{\{2\}(\{0\}+\{2\}+\{4\}+\ldots)\}_1 \\ &= \varepsilon^1[\{2\}+\{4\}+\{6\}+\ldots] \\ &= \{3\}+\{5\}+\{7\}+\ldots,\end{aligned}$$

while for $N \geqslant 2$

$$\langle 1(2)\rangle \downarrow \varepsilon^{1}\{(\{2\}-\{2^2\})D\}_2$$
$$= \varepsilon^{1}[\{2\}+\{31\}+\{4\}+\{42\}+\{51\}+\{6\}+\ldots]$$
$$= \{31^{N-1}\}+\{421^{N-2}\}+\{51^{N-1}\}+\{531^{N-2}\}+\{621^{N-2}\}+\{71^{N-1}\}+\ldots.$$

In a similar manner we have for $N \geqslant \sigma$ and $\lambda_1 \geqslant 2$

$$\langle 3/2(\lambda_1 1)\rangle \downarrow \varepsilon^{3/2}[[\{\lambda_1 1\}-\{\lambda_1 21\}]D]_3. \tag{3.11}$$

The results obtained for the $Sp(N, R) \downarrow U(N)$ branching rules can also be used to develop procedures for calculating the Kronecker products of the unirrep of $Sp(N, R)$ a subject we shall return to at a later time.

4. Signed sequences

In specifying all the inequivalent finite dimensional unirreps of $O(2\sigma)$ it is conventional following Littlewood (1940, p 227) to introduce standard labels by the identification

$$[\lambda] \equiv \begin{cases} [\mu] \text{ with } (\tilde{\mu}) = (\tilde{\mu}_1\tilde{\mu}_2\ldots) = (\tilde{\lambda}_1, \tilde{\lambda}_2\ldots) & \text{if } \tilde{\lambda}_1 \leqslant \sigma \\ [\mu]^* \text{ with } (\tilde{\mu}) = (2\sigma-\tilde{\lambda}_1, \tilde{\lambda}_2\ldots) & \text{if } \sigma < \tilde{\lambda}_1 \leqslant 2\sigma - \tilde{\lambda}_2 \end{cases} \tag{4.1}$$

where $[\mu]$ and $[\mu]^*$ are associated unirreps differing only by a factor of $\det A = \pm 1$ in their images of a group element A of $O(2\sigma)$. If 2σ is even and $\tilde{\mu}_1 = \sigma$ then $[\mu] = [\mu]^*$ and the unirrep is said to be self-associated.

The standard labels $[\mu]$ and $[\mu]^*$ for characters of $O(2\sigma)$ are restricted to partitions (μ) into not more than σ (or $\sigma - \frac{1}{2}$) parts. The equivalent standard labels $[\lambda]$, defined by (4.1), are restricted by $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leqslant 2\sigma$, showing an interesting parallel between the admissible $O(2\sigma)$ representations $[\lambda]$ and the $Sp(N, R)$ representations $\langle\sigma(\lambda)\rangle$ that are admissible by theorem 1.

In § 3 we encountered non-standard labels of $O(2\sigma)$ possessing up to 2σ parts and these must be modified to produce standard labels. In particular we need to know what sequence of non-standard labels will yield a given $[\lambda]$ ($\equiv[\mu]$ or $[\mu]^*$) of $O(2\sigma)$ upon application of the $O(2\sigma)$ modification rules. In this instance the modification rules for $O(2\sigma)$ as stated by Newell (1951) are most convenient.

The two infinite S-function series denoted by C and G (Black *et al* 1983) play a central role in Newell's analysis. They are defined

$$C = \{0\} + \sum_{\gamma} (-1)^{\omega_\gamma/2}\{\gamma\} \tag{4.2a}$$

$$G = \sum_{\varepsilon} (-1)^{(\omega_\varepsilon - r_\varepsilon)/2}\{\varepsilon\} \tag{4.2b}$$

where ω_γ and ω_ε are the weights of the partitions (γ) and (ε) respectively and r_γ, r_ε their corresponding ranks, defined for example in Wybourne (1970). (γ) is any partition in the Frobenius form

$$(\gamma) = \begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ a_1-1 & a_2-1 & \ldots & a_r-1 \end{pmatrix}$$

and (ε) is any self-conjugate partition.

Let C^{o} denote all odd-rank terms in the C-series and C^{e} all even rank terms. Likewise let G^{o} denote all odd-weight terms in G and G^{e} all even-weight terms. The first few terms of each series are as follows:

$$C^{\circ}=-\{2\}+\{31\}-\{41^2\}+\{51^3\}-\{61^4\}+\{4^3\}+\{71^5\}-\{81^6\}-\{54^21\}\ldots$$

$$C^{e}=\{0\}-\{3^2\}+\{431\}-\{531^2\}-\{4^22\}+\{5421\}-\{5^22^2\}\ldots$$

$$G^{\circ}=\{1\}-\{21\}+\{31^2\}-\{41^3\}-\{3^3\}-\{61^5\}+\{43^21\}+\{71^6\}-\{53^31^2\}-\{4^232\}\ldots$$

$$G^{e}=\{0\}-\{2^2\}+\{321\}-\{3^22\}-\{421^2\}+\{521^3\}+\{4321\}-\{621^4\}-\{5321^2\}-\{4^22^2\}\ldots.$$

Following Newell's (1951) results we can state:

(i) For σ an integer all characters $[\lambda]$ of $O(2\sigma)$ labelled by partitions having more than σ parts vanish except for those that can be re-expressed as standard labels via the equivalences

$$[\lambda_1\lambda_2\ldots\lambda_\sigma(C^{\circ})_\sigma]=[\lambda_1\lambda_2\ldots\lambda_\sigma]^* \tag{4.3a}$$

$$[\lambda_1\lambda_2\ldots\lambda_\sigma(C^{e})_\sigma]=[\lambda_1\lambda_2\ldots\lambda_\sigma]. \tag{4.3b}$$

Here the series C° and C^{e} are restricted to partitions of not more than σ parts.

(ii) For σ a half-integer all characters $[\lambda]$ of $O(2\sigma)$ labelled by partitions having more than $\sigma-\frac{1}{2}$ parts vanish except for those that can be re-expressed as standard labels via the equivalences

$$[\lambda_1\lambda_2\ldots\lambda_{\sigma-1/2}(G^{\circ})_{\sigma+1/2}]=[\lambda_1\lambda_2\ldots\lambda_{\sigma-1/2}]^* \tag{4.3c}$$

$$[\lambda_1\lambda_2\ldots\lambda_{\sigma-1/2}(G^{e})_{\sigma+1/2}]=[\lambda_1\lambda_2\ldots\lambda_{\sigma-1/2}]. \tag{4.3d}$$

It is important to note that in using (4.3*a*)-(4.3*d*), partitions which are not in standard descending order may arise and these must be rearranged using the *S*-function modification rules (Littlewood 1940, Wybourne 1970).

(I) In any *S*-function two consecutive parts may be interchanged provided that the preceding part is decreased by unity and the succeeding part is increased by unity, the *S*-functions being thereby changed in sign.

(II) In any *S*-function if any part exceed by unity the preceding part the value of the *S*-function is zero.

(III) The value of any *S*-function is zero if the last part is a negative number.

With the above provisos, equations (4.3*a*)-(4.3*d*) will rapidly lead to the determination of the complete sequence of non-standard labelled $O(2\sigma)$ characters that are related to a given standard labelled character for a given value of σ.

By way of example consider the associated irrep $[21]^*$ of O(6). We have from (4.3*a*) the signed sequence

$$-[2102],\ +[21031],\ -[21041^2],\ [2104^3]$$

reordering the above partitions gives the signed sequence

$$+[21^3],\ -[2^31^3]$$

since the second and fourth terms vanish. From (4.3*b*) we have associated with [21] the signed sequence

$$[21],\ -[2^51].$$

Since for O(6) $[21^2]^*\equiv[21^2]$, the sequence associated with $[21^2]$ will contain terms from both (4.3*a*) and (4.3*b*) giving the signed sequence

$$[21^2],\ -[2^41^2].$$

These results and others like them may also be arrived at and checked by the use of modification rules involving the addition of certain boundary hooks to Young diagrams specified by partitions (King 1975, Black *et al* 1983).

As a consequence of the above results, for each $\sigma=\sigma^{\max}$ and each $(\lambda)=(\lambda^{\min})$ satisfying (2.8) the corresponding character $[\lambda]$ ($=[\mu]$ or $[\mu]^*$) of $O(2\sigma)$ can be associated with a signed sequence (λ_s^σ of partitions which serve as non-standard $O(2\sigma)$ labels involving up to 2σ parts. The signed sequence λ_s^σ to be associated with a given unirrep $\sigma(\lambda)$ of $Sp(N, R)$ is found by relating (λ) to a standard irrep $[\mu]$ or $[\mu]^*$ through (4.1) and then using (4.3*a*)-(4.3*d*) to construct additional terms. Thus we obtain the branching rule (3.9) for all $Sp(N, R)$ representations that are admissible by theorem 1.

It follows, by construction, that every admissible $Sp(N, R)$ unirrep is realised in the space of some $2\sigma N$-dimensional harmonic oscillator. However, a given $Sp(N, R)$ unirrep $\langle(\tau)\rangle$ only occurs in the space of the $2\sigma N$-dimensional oscillator for a particular value of σ if $[\lambda]$, defined by (2.4), is a standard $O(2\sigma)$ label by (4.1); i.e. if $\tilde{\lambda}_1+\tilde{\lambda}_2\leqslant 2\sigma$. Thus, for example, the $Sp(2, R)$ unirrep

$$\langle(32)\rangle\equiv\langle 2(3)\rangle\equiv\langle 1(21)\rangle$$

occurs in the space of the eight-dimensional harmonic oscillator ($\sigma=2$) but not in the space of the four-dimensional harmonic oscillator ($\sigma=1$).

Thus for the $\langle 3(21)\rangle$ unirrep of $Sp(N, R)$ with $N>2$, we have $\sigma=3$ and hence the signed sequence

$$(21)_s^3=(21)-(2^5 1),$$

and for $\langle 3(21^3)\rangle$ with $N>4$

$$(21^3)_s^3=(21^3)-(2^3 1^3),$$

while for $\langle 3(21^2)\rangle$ with $N>3$

$$(21^2)_s^3=(21^2)-(2^4 1^2).$$

In general the signed sequence will be of the form

$$\lambda_s^\sigma=\sum_\nu g_\lambda^\nu(\nu) \tag{4.4}$$

where the summation is over all relevant $O(2\sigma)$ partition labels involving up to 2σ parts and the coefficients g_λ^ν are either 0 or ± 1. The leading term in the sequence λ_s^σ will be (λ). The second term in the sequence will have at least $2\sigma-\tilde{\mu}_1+1$ parts. The signed sequences for $\sigma\leqslant 5/2$ are given in table 1.

5. Sp(*N*, *R*) matrix elements

From the coherent state theory of the symplectic groups (Rowe 1984), one obtains a non-unitary realisation of the $Sp(N, R)$ algebra in the form

$$\Gamma(A_{ij})=[\Lambda, a_{ij}^\dagger], \qquad \Gamma(B_{ij})=a_{ij}$$
$$\Gamma(C_{ij})=\mathbb{C}_{ij}+(a^\dagger a)_{ij} \tag{5.1}$$

where

$$(a_{ij}^\dagger, a_{ij}; i, j=1, \ldots, N)$$

are symmetric boson operators ($a_{ij} = a_{ji}$), satisfying

$$[a_{ij}, a^\dagger_{lk}] = \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}, \tag{5.2}$$

and the ($\mathbb{C}_{ij}$) are a basis for a U(N) algebra

$$[\mathbb{C}_{ij}, \mathbb{C}_{lk}] = \delta_{jl}\mathbb{C}_{ik} - \delta_{ik}\mathbb{C}_{lj} \tag{5.3}$$

that commutes with the bosons; i.e.

$$[\mathbb{C}_{ij}, a^\dagger_{lk}] = [\mathbb{C}_{ij}, a_{lk}] = 0. \tag{5.4}$$

Λ is a U(N) invariant operator, given by

$$\Lambda = \tfrac{1}{2}\mathrm{Tr}[(\mathbb{C} + a^\dagger a)(\mathbb{C} + a^\dagger a)] - \tfrac{1}{4}\mathrm{Tr}(a^\dagger a a^\dagger a) - \tfrac{1}{4}(N+1)\,\mathrm{Tr}(a^\dagger a) \tag{5.5}$$

where we use matrix notation; e.g. $\mathrm{Tr}(a^\dagger a) = \Sigma_{ij}\, a^\dagger_{ij} a_{ji}$.

The fact that Γ is indeed a realisation can be ascertained directly by checking that it satisfies all the Sp(N, R) commutation relations (2.2). Thus we may dispense with any limitations implicit in the coherent state origin of (5.1).

Let $V_{\mathrm{u}}^{\sigma(\lambda)}$ be the carrier space for a U(N) ⊃ U(1) × SU(N) unirrep $\sigma(\lambda)$ and let V_{w} be the carrier space for a representation of the Weyl (boson) algebra. The product space

$$V_{\mathrm{uw}}^{\sigma(\lambda)} = V_{\mathrm{u}}^{\sigma(\lambda)} \times V_{\mathrm{w}} \tag{5.6}$$

then carries a unirrep of the direct product unitary-Weyl algebra. Furthermore, if $|\sigma(\lambda)LW\rangle$ is the U(N) lowest weight state and $|0\rangle$ is the boson vacuum, then

$$|\sigma(\lambda)LW) = |\sigma(\lambda)LW\rangle|0\rangle \tag{5.7}$$

is the lowest weight state for the unitary-Weyl algebra.

The boson raising operator $a^\dagger$ is clearly a U(N) tensor of rank (2), WRT the realisation Γ of U(N) ⊂ Sp(N, R), defined by (5.1). Let $\chi^{(n)}(a^\dagger)$ be a suitably normalised polynomial in the raising operators of tensor rank (n), where (n) is a partition with even parts; i.e. $\{n\} \in D$. Then an orthonormal basis for $V_{\mathrm{uw}}^{\sigma(\lambda)}$ is given by states of the form

$$|\sigma(\lambda)n\delta\omega\alpha) = [\chi^{(n)}(a^\dagger)|\sigma(\lambda))]^{\delta\omega}_{\alpha} \tag{5.8}$$

where δ is a multiplicity index and α labels a basis for the coupled U(N) unirrep (ω).

The U(N)-invariant operator Λ is conveniently diagonal in this basis with eigenvalues

$$\Omega(\sigma n\omega) = \tfrac{1}{4}\sum_{i=1}^{N} [2\omega_i^2 - n_i^2 + 2(N+1)(\omega_i - n_i) - 2i(2\omega_i - n_i)]. \tag{5.9}$$

Now observe that the lowest-weight state $|\sigma(\lambda)\mathrm{Lw}\rangle$ is also a lowest weight state for the realisation Γ of the Sp(N, R) algebra. It follows that Sp(N, R) acts irreducibly on the subspace $V_{\mathrm{sp}}^{\sigma(\lambda)} \subset V_{\mathrm{uw}}^{\sigma(\lambda)}$ generated from the lowest-weight state $|\sigma(\lambda)LW\rangle$ by the $\Gamma(A)$ raising operators. A basis for $V_{\mathrm{sp}}^{\sigma(\lambda)}$ is obtained by eliminating from the $V_{\mathrm{uw}}^{\sigma(\lambda)}$ basis, (5.8), all states for which

$$[\chi^{(n)}(\Gamma(A))|\sigma(\lambda))]^{\delta\omega}_{\alpha} = 0. \tag{5.10}$$

If $\langle\sigma(\lambda)\rangle_{\mathrm{uw}}$ is the unitary-Weyl character for the unirrep $\sigma(\lambda)$, defined above, then from its construction, its U(N) content is given by

$$\langle\sigma(\lambda)\rangle_{\mathrm{uw}} \downarrow \varepsilon^{\sigma}\{\lambda D\}_N. \tag{5.11}$$

It follows that the removal of the redundant U(N) subspaces to yield an irreducible Sp(N, R) representation space, corresponds precisely to the modification of the branching rule, for $\sigma < N$ given by equation (3.10).

For $\sigma < N$, the representation of Sp(N, R) carried by $V_{\rm uw}^{\sigma(\lambda)}$ is an example of a representation that is reducible but not fully reducible. The restriction to $V_{\rm sp}^{\sigma(\lambda)}$ gives a fully reduced representation.

To simplify notation, let a single index $i \equiv (\sigma n \delta)$ distinguish multiply occurring U(N) unirreps in $V_{\rm uw}^{\sigma(\lambda)}$.

Theorem 2. If

$$\Omega(i\omega') - \Omega(j\omega) = 0$$

for all $|j\omega\rangle \in V_{\rm sp}^{\sigma(\lambda)}$ for which the SU(N) reduced matrix element

$$(i\omega' \| a^\dagger \| j\omega) \neq 0$$

then the state $|i\omega'\rangle$ is not in $V_{\rm sp}^{\sigma(\lambda)}$.

Proof. Observe that $|i\omega'\rangle \in V_{\rm sp}^{\sigma(\lambda)}$ if and only if

$$(i\omega' \| \Gamma(A) \| j\omega) = [\Omega(i\omega') - \Omega(j\omega)](i\omega' \| a^\dagger \| j\omega) \tag{5.12}$$

does not vanish for some $|j\omega\rangle \in V_{\rm sp}^{\sigma(\lambda)}$.

The difference

$$\Delta\Omega(\sigma n'\omega'; n\omega) = \Omega(\sigma n'\omega') - \Omega(\sigma n\omega) \tag{5.13}$$

is evalutated directly from (5.9). Two situations occur

$$\text{(i)} \qquad \begin{aligned} &\omega_i' = \omega_i + 2, \qquad n_k' = n_k + 2, \\ &\Delta\Omega = 2\omega_i - n_k - 2i + k + 1 \end{aligned} \tag{5.14}$$

and

$$\text{(ii)} \qquad \begin{aligned} &\omega_i' = \omega_i + 1, \qquad \omega_j' = \omega_j + 1, \qquad i \neq j, \qquad n_k' = n_k + 2, \\ &\Delta\Omega = \omega_i + \omega_j - n_k - i - j + k. \end{aligned} \tag{5.15}$$

It is significant that $\Delta\Omega$, unlike Ω, does not depend on N. The utility of theorem 2 can now be illustrated. Consider an Sp(N, R) unirrep $\sigma(\lambda_1 1)$. For $\sigma \geq N$, (3.10) gives

$$\{\sigma(\lambda_1 1)\rangle \downarrow \varepsilon^\sigma(\{\lambda_1 1\}) + \{\lambda_1 + 2, 1\} + \{\lambda_1 + 1, 2\} + \{\lambda_1 + 1, 1^2\} + \{\lambda_1, 3\} + \{\lambda_1 21\} + \ldots).$$

Now the $\{\lambda_1 21\}$ component corresponds to the state

$$|\sigma(\lambda_1 1)(2)\omega'\rangle \qquad \text{with } \omega_1' = \lambda_1 + \sigma,\ \omega_2' = \sigma + 2,\ \omega_3' = \sigma + 1.$$

It can be reached by a raising operator only from the $\{\lambda_1 1\}$ state, for which $\omega_1 = \lambda_1 + \sigma$, $\omega_2 = \sigma + 1$, $\omega_3 = \sigma$. For these two states

$$\Delta\Omega = 2\sigma - 3$$

vanishes for $\sigma = \frac{3}{2}$. Thus one understands the removal of the contribution of this state, for $\sigma = \frac{3}{2}$, in (3.11).

We now seek a test for unitarity. In the analysis of Rowe (1984), it was assumed *a priori* that the Sp(N, R) representation $\sigma(\lambda)$ was equivalent to a unitary representation. Here, however, where we start with the non-unitary realisation (5.1), we have no guarantee of this.

It nevertheless follows from Rowe (1984) that, if the Sp(N, R) unirrep $\sigma(\lambda)$ is equivalent to a unitary representation, we may make a transformation

$$\gamma(X)=\kappa^{-1}\Gamma(X)\kappa, \qquad X\in \mathrm{Sp}(N, R), \tag{5.16}$$

with $\kappa=\kappa^{\dagger}$ Hermitian and U(N) invariant, such that the action of γ is unitary. The equation $\gamma(B)^{\dagger}=\gamma(A)$, required for unitarity, then implies that κ satisfies

$$\kappa^{2}a^{\dagger}\kappa^{-2}=[\Lambda, a^{\dagger}]. \tag{5.17}$$

Hence one has

$$\kappa^{2}\sum_{ij}a^{\dagger}_{ij}a_{ji}=\sum_{ij}[\Lambda, a^{\dagger}_{ij}]\kappa^{2}a_{ji}$$

from which one derives the recursion relation for the matrix elements of κ^{2}

$$\langle i\omega|\kappa^{2}|j\omega)=\frac{2}{N(j)}\sum_{kl\omega'\omega}\Delta\Omega(i\omega, k\omega')(k\omega'|\kappa^{2}|l\omega')(i\omega\|a^{\dagger}\|k\omega')(j\omega\|a^{\dagger}\|l\omega')^{*} \tag{5.18}$$

where

$$N(l)=\sum_{ij}(l\omega|a^{\dagger}_{ij}a_{ji}|l\omega)=\sum_{k}n_{k}(l)$$

and where, in the definition of the SU(N) reduced matrix elements, the SU(N) tensors $a^{\dagger}$ and a are now normalised in the standard way

$$[a_{\mu}, a^{\dagger}_{\nu}]=\delta_{\mu\nu} \qquad \mu, \nu=1,\ldots,\tfrac{1}{2}N(N+1).$$

Theorem 3. If the Sp(N, R) representation $\sigma(\lambda)$ is unitary and $\Delta\Omega(\omega', \omega)=\Omega(i\omega')-\Omega(j\omega)$ is non-vanishing and independent of any multiplicity indices and if

$$(i\omega'\|a^{\dagger}\|j\omega)\neq 0$$

then

$$\Delta\Omega(\omega', \omega)>0.$$

Proof. From (5.17) and the independence of $\Delta\Omega(\omega', \omega)$ on the multiplicity indices, we infer

$$\sum_{k,j}(j\omega\|a\|i\omega')(i\omega'\|\kappa^{2}a^{\dagger}\kappa^{-2}\|j\omega)=\Delta\Omega(\omega', \omega)\sum_{ij}|(i\omega'\|a^{\dagger}\|j\omega)|^{2}.$$

The LHS can be re-expressed

$$\mathrm{LHS}=\sum_{i,j}|(i\omega'\|\kappa a^{\dagger}\kappa^{-1}\|j\omega)|^{2},$$

showing that, under the conditions of the theorem, $\Delta\Omega(\omega', \omega)$ is strictly positive.

Proof of theorem 1. First observe that any Sp(N, R) irrep of the harmonic series is necessarily unitary because, in terms of the harmonic oscillator raising and lowering (boson) operators

$$(b^{\dagger}_{\nu i}, b_{\nu i};\ \nu=1,\ldots,2\sigma,\ i=1,\ldots,N),$$

the Sp(N, R) algebra is realised.

$$A_{ij}=\sum_{\nu=1}^{2\sigma} b^{\dagger}_{\nu i}b^{\dagger}_{\nu j} \qquad B_{ij}=\sum_{\nu=1}^{2\sigma} b_{\nu i}b_{\nu j}$$
$$C_{ij}=\tfrac{1}{2}\sum_{\nu=1}^{2\sigma}(b^{\dagger}_{\nu i}b_{\nu j}+b_{\nu j}b^{\dagger}_{\nu i}), \tag{5.19}$$

which is manifestly unitary. Since, as observed in § 4, the harmonic series contains all the (D_+-series metaplectic) representations admissible by theorem 1, it follows that all admissible representations are unitary.

It remains to show that a representation $\sigma(\lambda)$ that is not admissible is not unitary. We put $\sigma=\sigma^{\max}$ and $(\lambda)=(\lambda^{\min})$ and consider the following two cases.

(i) $\tilde{\lambda}_1=\tilde{\lambda}_2=r.$

The lowest weight U(N) state $\omega=\sigma(\lambda)$ has

$$\omega_r=\sigma+\lambda_r, \qquad \omega_{r+1}=\sigma, \qquad n_1=0.$$

(Recall that $r\leqslant N-1$ for $\sigma=\sigma^{\max}$). If we evaluate $\Delta\Omega(\omega',\omega)$ for

$$\omega'_{r+1}=\omega_{r+1}+2, \qquad n'_1=2,$$

we obtain from (5.14),

$$\Delta\Omega(\omega',\omega)=2\sigma-2r=2\sigma-\tilde{\lambda}_1-\tilde{\lambda}_2$$

which is negative for $\tilde{\lambda}_1+\tilde{\lambda}_2>2\sigma$ violating (2.8).

(ii) $\tilde{\lambda}_1=r>\tilde{\lambda}_2=s.$

The lowest weight U(N) state $\omega=\sigma(\lambda)$ has

$$\omega_{s+1}=\sigma+1, \qquad \omega_{r+1}=\sigma, \qquad n_1=0.$$

For

$$\omega'_{s+1}=\sigma+2, \qquad \omega'_{r+1}=\sigma+1, \qquad n'_1=2,$$

we obtain from (5.15)

$$\Delta\Omega(\omega',\omega)=2\sigma-r-s=2\sigma-\tilde{\lambda}_1-\tilde{\lambda}_2$$

which is negative for $\tilde{\lambda}_1+\tilde{\lambda}_2>2\sigma$, again violating (2.8).

Thus by theorem 2 we require $\tilde{\lambda}_1+\tilde{\lambda}_2\leqslant 2\sigma$ for unitarity as stated in theorem 1.

For a unitary representation, (5.18) is easily solved for κ, as illustrated in Rowe (1984), to obtain the matrix elements of

$$\gamma(A)=\kappa a^{\dagger}\kappa^{-1}. \tag{5.20}$$

In particular, for multiplicity free states, (5.17) gives immediately

$$\left[\frac{\kappa(\omega')}{\kappa(\omega)}\right]^2(\omega'\|a^{\dagger}\|\omega)=\Delta\Omega(\omega',\omega)(\omega'\|a^{\dagger}\|\omega). \tag{5.21}$$

Thus, since $\Delta\Omega$ is positive definite, by theorem 3, we obtain the analytic expression

$$(\omega'\|\gamma(A)\|\omega)=[\Delta\Omega(\omega',\omega)]^{1/2}(\omega'\|a^{\dagger}\|\omega). \tag{5.22}$$

Explicit analytic expressions for matrix elements were given previously (Castaños *et al* 1984, Rowe *et al* 1984, Deenen and Quesne 1984, Rowe 1985) for the $\sigma(0)$ class of Sp(3, R) unirreps. We now extend them to any Sp(N, R) unirrep of the form $\sigma(1^\alpha)$, $0 \leqslant \alpha \leqslant N$. These unirreps are all multiplicity free with states labelled uniquely by their U(N) labels (ω) with

$$\omega_i = \sigma + n_i \qquad \text{or} \qquad \omega_i = \sigma + n_i + 1, \qquad i = 1, \ldots, N,$$

where (n) is a partition of even parts and (ω) has α odd parts. Three kinds of matrix element occur

$$\text{(i)} \qquad \omega_i = \sigma + n_i, \qquad \omega_i' = \omega_i + 2. \tag{5.23a}$$

Since ω_i and ω_i' are both even, we require $n_i' = n_i + 2$ and (5.14) gives

$$\Delta\Omega(\omega', \omega) = 2\sigma + n_i - i + 1. \tag{5.23b}$$

$$\text{(ii)} \qquad \omega_i = \sigma + n_i + 1, \qquad \omega_i' = \omega_i + 2. \tag{5.24a}$$

Again $n_i' = n_i + 2$ and

$$\Delta\Omega(\omega', \omega) = 2\sigma + n_i - i + 3. \tag{5.24b}$$

$$\text{(iii)} \qquad \omega_i = \sigma + n_i + 1, \qquad \omega_j = \sigma + n_j,$$

$$\omega_i' = \omega_i + 1, \qquad \omega_j' = \omega_j + 1, \qquad i \neq j. \tag{5.25a}$$

Since $\omega_i' = \sigma + n_i + 2$, it follows that $n_i' = n_i + 2$ and

$$\Delta\Omega(\omega', \omega) = 2\sigma + n_j - j + 1. \tag{5.25b}$$

Thus, the Sp(N, R) matrix elements for the $\sigma(1^\alpha)$ unirreps are obtained explicitly in terms of much simpler boson matrix elements by (5.22). The $N = 3$ boson matrix elements were evaluated for the $\sigma(0)$ unirreps by Quesne (1981) and for arbitrary $\sigma(\lambda)$ by Rosensteel and Rowe (1983).

For arbitrary Sp(N, R) unirreps, many of the states are multiplicity free and Sp(N, R) matrix elements for such states are also given analytically by (5.22). The multiplicities are given by the branching rules of § 4. For example, for $\sigma(2)$ and $\sigma \geqslant N$, one has, by (3.10),

$$\langle\sigma(2)\rangle = \varepsilon^\sigma(\{2\} + \{4\} + \{2^2\} + \{6\} + 2\{42\} + \ldots),$$

and one sees that the first multiplicity occurs for $\{42\}$, i.e. for $(\omega) = (4+\sigma, 2+\sigma, \sigma, \ldots, \sigma)$.

For any unirrep $\sigma(\lambda)$, the stretched states $(\omega) = (\lambda_1 + \sigma + n_1, \lambda_2 + \sigma_1, \lambda_3 + \sigma, \ldots)$ are always multiplicity free. This is an important class of substates of major interest in the theory of nuclear collective motion (Arickx *et al* 1979, Park *et al* 1984). It is significant therefore that (5.22) gives analytic expressions for stretched matrix elements.

It is worth remarking that the branching rules for compact Lie groups involve the formation of skew-S functions (or equivalently S-function division) giving rise to a finite number of terms. In this paper we have presented what we believe is the first formulation of a branching rule for a non-compact group in terms of S-functions. In this case the S-functions appear as a non-terminating infinite sequence as would be expected for a non-compact group.

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